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Gauge groups of noncommutative principal bundles

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-In this talk I would like to present a study of the notions of gauge group in NC geometry. Besides the NC geometry of the phase-space of quantum mechanics NC geometry in physics is present in the study of gauge theories in NC space, these capture a sector of string theory and arise in the study of T-duality.

-The study of NC gauge groups can be useful to a better/deeper understanding of gauge theories.

-Gauge theory on M with gauge group G, gauge transformations are maps M-¿ G. This is true locally.

-The geometric picture is that of a principal G-bundle $P->M$

The gauge group are then the automorphisms of P

We have to understand this notion in the NC context.

-NC Principal bundle. This is Hopf-Galois extension

A **braided Lie algebra** associated with a triangular Hopf algebra (K, R) is a rep. g of K with

$$[\ ,\ ] : \mathrm{g} \otimes \mathrm{g} \to \mathrm{g}$$

such that

(i) K-equivariance: $k \triangleright [u, v] = [k_{(1)} \triangleright u, k_{(2)} \triangleright v]$,

(ii) braided antisymmetry: $[u, v] = -[\mathsf{R}_\alpha \triangleright v, \mathsf{R}^\alpha \triangleright u]$,

(iii) braided Jacobi identity: $[u, [v, w]] = [[u, v], w] + [\mathsf{R}_\alpha \triangleright v, [\mathsf{R}^\alpha \triangleright u, w]]$,

The associated UEA $U(\mathrm{g})$ is a braided Hopf Algebra.

Braided derivations of A:

$$\mathsf{Der}A := \{\psi : A \to A;\, \psi(aa') = \psi(a)a' + (\mathsf{R}_\alpha \triangleright a)\,(\mathsf{R}^\alpha \triangleright \psi)(a')\}$$

form a braided Lie algebra.

If A is a braided commutative algebra:

$$a\, a' = (\mathsf{R}_\alpha \triangleright a')\,(\mathsf{R}^\alpha \triangleright a)\ ,$$

then $a\psi$ is a braided derivation if ψ is, hence $\mathrm{Der}A$ is an A-module.
Moreover, we have compatibility with [,]:

$$[u, av] = u(a)v + \mathsf{R}_\alpha \triangleright a[\mathsf{R}^\alpha \triangleright u, v] \tag{1}$$

Braided Lie Algebra of Gauge Transformations

Def. Let (K, R) be a triangular Hopf algebra and $B = A^{coH} \subseteq A$ a K-equivariant Hopf-Galois extension. Infinitesimal gauge transformations are the H-equivariant, braided vertical derivations

$$\begin{aligned}\mathrm{aut}_B^{\mathsf{R}}(A) := \{u : A \to A \mid\ & \delta(u(a)) = u(a_{(0)}) \otimes a_{(1)},\\ & u(aa') = u(a)a' + (\mathsf{R}_\alpha \triangleright a)(\mathsf{R}^\alpha \triangleright u)(a')\ ,\\ & u(b) = 0,\ \text{for all } a, a' \in A, b \in B\}\ .\end{aligned}$$

Prop. $\mathrm{aut}_B^{\mathsf{R}}(A)$ is a K-braided Lie algebra.
The UEA $U(\mathrm{aut}_B^{\mathsf{R}}(A))$ is a braided Hopf Algebra.
If A is quasi commutative then eq. (1) holds.

Quantum homogenous space example.

Theorem A a Hopf algebra. $A \to H$ Hopf algebra projection. Let the dual Hopf algebra H° be triangular and induce triangular structure on $K := A^\circ$. Then, the Hopf-Galois extension $B = A^{coH} \subseteq A$ is quasi-commutative and infinitesimal gauge transformations are the vertical braided vector fields in

$$B \otimes \mathsf{Der}^{\mathfrak{R}}(A)_{\mathsf{inv}} \; ,$$

where $\mathsf{Der}^{\mathfrak{R}}(A)_{\mathsf{inv}}$ are the right-invariant vector fields defining the bicovariant differential calculus on (A, R).

Examples from Drinfeld Twists deformation

Drinfeld Twist deformation of principal bundles give NC principal bundles (Hopf-Galois extensions)

The deformation of the gauge Lie algebra gives the braided gauge Lie algebra.

Example . *Trivial bundle.* $B \subseteq B \otimes H$, with $B = \mathcal{O}(M)$ and $H = \mathcal{O}(G)$

Infinitesimal gauge transformations is the free B-module

$$\mathsf{aut}_B(B \otimes H) = B \otimes \mathsf{Lie(G)} \tag{3}$$

with $\mathsf{Lie(G)}$ the right-invariant vector fields on G. Let X^i basis of $\mathsf{Lie(G)}$.

Lie bracket: $[b_i X^i, b'_j X^j] = b_i b'_j [X^i, X^j]$.

Twist deform with an abelian twist theta

$$\mathsf{F} = e^{\pi i \theta (H_1 \otimes H_2 - H_2 \otimes H_1)} \tag{4}$$

associated with a torus action on B and obtain

$$[b_i \bullet_\theta X^i, b'_j \bullet_\theta X^j]_{\mathsf{R_F}} = b_i \bullet_\theta b'_j \bullet_\theta [X^i, X^j],$$

where $(b_i \bullet_\theta X^i)(a) = b_i \bullet_\theta X^i(a)$ for all $a \in (B_\theta \otimes H)$.

Example . *The instanton bundle on the sphere S^4_θ.* The spheres S^7 and S^4 are homogeneous spaces.

Lie algebra of $so(5)$

$$[H_1, H_2] = 0 \; ; \quad [H_j, E_{\mathrm{r}}] = r_j E_{\mathrm{r}} \; ;$$
$$[E_{\mathrm{r}}, E_{-\mathrm{r}}] = r_1 H_1 + r_2 H_2 \; ; \quad [E_{\mathrm{r}}, E_{\mathrm{s}}] = N_{rs} E_{\mathrm{r+s}} \; .$$

H_j, $j = 1, 2$ generators of Cartan subalgebra, E_{r} labelled by the eight roots $\mathrm{r} = (r_1, r_2)$.
Infinitesimal gauge transformations are H-equivariant derivations X which are vertical: $X(b) = 0$ for $b \in \mathcal{O}(S^4)$. They form the $\mathcal{O}(S^4)$-module generated by

$$\begin{aligned}
K_1 &= 2xH_2 + \beta^*\sqrt{2}E_{01} + \beta\sqrt{2}E_{0-1} & K_2 &= 2xH_1 + \alpha^*\sqrt{2}E_{10} + \alpha\sqrt{2}E_{-10} \\
W_{01} &= \sqrt{2}\big(\beta H_1 + \alpha^* E_{11} + \alpha E_{-11}\big) & W_{0-1} &= \sqrt{2}\big(\beta^* H_1 + \alpha^* E_{1-1} + \alpha E_{-1-1}\big) \\
W_{10} &= \sqrt{2}\big(\alpha H_2 - \beta^* E_{11} + \beta E_{1-1}\big) & W_{-10} &= \sqrt{2}\big(\alpha^* H_2 + \beta^* E_{-11} - \beta E_{-1-1}\big) \\
W_{11} &= 2xE_{11} + \alpha\sqrt{2}E_{01} - \beta\sqrt{2}E_{10} & W_{-1-1} &= 2xE_{-1-1} + \alpha^*\sqrt{2}E_{0-1} - \beta^*\sqrt{2}E_{-10} \\
W_{1-1} &= -2xE_{1-1} + \beta^*\sqrt{2}E_{10} + \alpha\sqrt{2}E_{0-1} & W_{-11} &= -2xE_{-11} + \beta\sqrt{2}E_{-10} + \alpha^*\sqrt{2}E_{01}
\end{aligned}$$

where $\alpha^*\alpha + \beta^*\beta + x^2 = 1$ coordinates of $\mathcal{O}(S^4)$.

Infinitesimal gauge transf.: $X = b_1 K_1 + b_2 K_2 + \sum_{\mathrm{r}} b_{\mathrm{r}} W_{\mathrm{r}}$
$b_1, b_2, b_{\mathrm{r}} \in \mathcal{O}(S^4)$.

Twist with (4) and obtain the $\mathcal{O}(SU(2))$-Hopf–Galois extension (θ-instanton bundle)

$$\mathcal{O}(S^4_\theta) = \mathcal{O}(S^7_\theta)^{co\mathcal{O}(SU(2))} \subset \mathcal{O}(S^7_\theta)$$

$\mathcal{O}(S^4_\theta)$ relations: $\alpha \bullet_\theta \alpha^* + \beta \bullet_\theta \beta^* + x \bullet_\theta x = 1$ with

$$\alpha \bullet_\theta \beta = e^{-2\pi i\theta} \beta \bullet_\theta \alpha \ , \quad \alpha \bullet_\theta \beta^* = e^{2\pi i\theta} \beta^* \bullet_\theta \alpha \tag{7}$$

The braided Lie algebra

$$\mathsf{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$$

of infinitesimal gauge transf. is generated, as an $\mathcal{O}(S^4_\theta)$-module, by the elements

$$\widetilde{X} = b_1 \bullet_\theta \widetilde{K}_1 + b_2 \bullet_\theta \widetilde{K}_2 + \sum_{\mathsf{r}} b_{\mathsf{r}} \bullet_\theta \widetilde{W}_{\mathsf{r}} \ .$$

The braided Lie bracket of generic elements $\widetilde{X}, \widetilde{X}'$ in $\mathsf{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ and $b, b' \in \mathcal{O}(S^4_\theta)$ is given by

$$[b \bullet_\theta \widetilde{X}, b' \bullet_\theta \widetilde{X}']_{\mathsf{R_F}} \ = \ b \bullet_\theta (\mathsf{R}_\alpha \triangleright b') \bullet_\theta [\mathsf{R}^\alpha \triangleright \widetilde{X}, \widetilde{X}']_{\mathsf{R_F}} \tag{8}$$

Independent braided Lie algebra brackets:

$$[\widetilde{K}_1, \widetilde{K}_2]_{\mathsf{R_F}} = \sqrt{2}(\alpha^* \bullet_\theta \widetilde{W}_{10} - \alpha \bullet_\theta \widetilde{W}_{-10})$$

$$[\widetilde{K}_1, \widetilde{W}_{01}]_{\mathsf{R_F}} = -\sqrt{2}\beta \bullet_\theta \widetilde{K}_2 + 2x \bullet_\theta \widetilde{W}_{01}$$

$$[\widetilde{K}_1, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = -2x \bullet_\theta \widetilde{W}_{1-1} + \sqrt{2}e^{\pi i\theta}\beta^* \bullet_\theta \widetilde{W}_{10}$$

$$[\widetilde{K}_1, \widetilde{W}_{10}]_{\mathsf{R_F}} = \sqrt{2}e^{-\pi i\theta}\beta \bullet_\theta \widetilde{W}_{1-1} - \sqrt{2}e^{\pi i\theta}\beta^* \bullet_\theta \widetilde{W}_{11}$$

$$[\widetilde{K}_1, \widetilde{W}_{11}]_{\mathsf{R_F}} = 2x \bullet_\theta \widetilde{W}_{11} - \sqrt{2}e^{-\pi i\theta}\beta \bullet_\theta \widetilde{W}_{10}$$

$$[\widetilde{K}_2, \widetilde{W}_{01}]_{\mathsf{R_F}} = \sqrt{2}e^{-\pi i\theta}\alpha^* \bullet_\theta \widetilde{W}_{11} + \sqrt{2}e^{\pi i\theta}\alpha \bullet_\theta \widetilde{W}_{-11}$$

$$[\widetilde{K}_2, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = 2x \bullet_\theta \widetilde{W}_{1-1} - \sqrt{2}e^{-\pi i\theta}\alpha \bullet_\theta \widetilde{W}_{0-1}$$

$$[\widetilde{K}_2, \widetilde{W}_{10}]_{\mathsf{R_F}} = 2x \bullet_\theta \widetilde{W}_{10} - \sqrt{2}\alpha \bullet_\theta \widetilde{K}_1$$

$$[\widetilde{K}_2, \widetilde{W}_{11}]_{\mathsf{R_F}} = 2x \bullet_\theta \widetilde{W}_{11} + \sqrt{2}e^{\pi i\theta}\alpha \bullet_\theta \widetilde{W}_{01}$$

$$[\widetilde{W}_{01}, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = \sqrt{2}\beta \bullet_\theta \widetilde{W}_{1-1} + \sqrt{2}e^{\pi i\theta}\alpha \bullet_\theta (\widetilde{K}_2 - \widetilde{K}_1)$$

$$[\widetilde{W}_{01}, \widetilde{W}_{10}]_{\mathsf{R_F}} = \sqrt{2}\beta \bullet_\theta \widetilde{W}_{10} - \sqrt{2}e^{\pi i\theta}\alpha \bullet_\theta \widetilde{W}_{01}$$

$$[\widetilde{W}_{01}, \widetilde{W}_{11}]_{\mathsf{R_F}} = \sqrt{2}\beta \bullet_\theta \widetilde{W}_{11}$$

$$[\widetilde{W}_{1-1}, \widetilde{W}_{10}]_{\mathsf{R_F}} = \sqrt{2}\alpha \bullet_\theta \widetilde{W}_{1-1}$$

$$[\widetilde{W}_{1-1}, \widetilde{W}_{11}]_{\mathsf{R_F}} = -\sqrt{2}e^{-\pi i\theta}\alpha \bullet_\theta \widetilde{W}_{10}$$

$$[\widetilde{W}_{10}, \widetilde{W}_{11}]_{\mathsf{R_F}} = \sqrt{2}\alpha \bullet_\theta \widetilde{W}_{11}$$

$$[\widetilde{W}_{-1-1}, \widetilde{W}_{01}]_{\mathsf{R_F}} = \sqrt{2}e^{2\pi i\theta}\beta \bullet_\theta \widetilde{W}_{-1-1} - \sqrt{2}e^{\pi i\theta}\alpha^* \bullet_\theta (\widetilde{K}_1 + \widetilde{K}_2)$$
$$[\widetilde{W}_{-1-1}, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = \sqrt{2}e^{-2\pi i\theta}\beta^* \bullet_\theta \widetilde{W}_{0-1}$$
$$[\widetilde{W}_{-1-1}, \widetilde{W}_{10}]_{\mathsf{R_F}} = \sqrt{2}\alpha \bullet_\theta \widetilde{W}_{-1-1} + \sqrt{2}e^{\pi i\theta}\beta^* \bullet_\theta (\widetilde{K}_1 + \widetilde{K}_2)$$
$$[\widetilde{W}_{-1-1}, \widetilde{W}_{11}]_{\mathsf{R_F}} = -2x \bullet_\theta (\widetilde{K}_1 + \widetilde{K}_2) - \sqrt{2}\alpha \bullet_\theta \widetilde{W}_{-10} - \sqrt{2}\beta \bullet_\theta \widetilde{W}_{0-1}$$
$$[\widetilde{W}_{-10}, \widetilde{W}_{01}]_{\mathsf{R_F}} = \sqrt{2}e^{2\pi i\theta}\beta \bullet_\theta \widetilde{W}_{-10} - \sqrt{2}\alpha^* \bullet_\theta \widetilde{W}_{01}$$
$$[\widetilde{W}_{-10}, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = -\sqrt{2}\alpha^* \bullet_\theta \widetilde{W}_{1-1} + \sqrt{2}e^{-\pi i\theta}\beta^* \bullet_\theta (\widetilde{K}_2 - \widetilde{K}_1)$$
$$[\widetilde{W}_{10}, \widetilde{W}_{-10}]_{\mathsf{R_F}} = \sqrt{2}(\beta^* \bullet_\theta \widetilde{W}_{01} + \beta \bullet_\theta \widetilde{W}_{0-1})$$
$$[\widetilde{W}_{-11}, \widetilde{W}_{01}]_{\mathsf{R_F}} = \sqrt{2}e^{2\pi i\theta}\beta \bullet_\theta \widetilde{W}_{-11}$$
$$[\widetilde{W}_{0-1}, \widetilde{W}_{01}]_{\mathsf{R_F}} = \sqrt{2}(\alpha^* \bullet_\theta \widetilde{W}_{10} - \alpha \bullet_\theta \widetilde{W}_{-10})$$
$$[\widetilde{W}_{-11}, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = 2x \bullet_\theta (\widetilde{K}_1 - \widetilde{K}_2) - \sqrt{2}\beta^* \bullet_\theta \widetilde{W}_{01} + \sqrt{2}\alpha \bullet_\theta \widetilde{W}_{-10}$$

NC Atiyah sequences

In classical case:

$$0 \to \mathsf{aut}_B(A) \to \mathsf{Der}_{\mathsf{inv}}A \to \mathsf{Der}B \to 0 \tag{9}$$

is an exact sequence of Lie algebras associated with the principal bundle $B = A^{coH} \subseteq A$. Let it be K-equivariant. Associated with the K_{F}-equivariant Hopf–Galois extension $B_{\mathsf{F}} = A_{\mathsf{F}}^{coH} \subseteq A_{\mathsf{F}}$ we have the exact sequence

$$0 \to \mathsf{aut}_{B_{\mathsf{F}}}^{\mathsf{R_F}}(A_{\mathsf{F}}) \to \mathsf{Der}_{\mathsf{inv}}A_{\mathsf{F}} \to \mathsf{Der}B_{\mathsf{F}} \to 0\,. \tag{10}$$